

# Remarks on the Prandtl Equation

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## Abstract

In a recent result of Gérard-Varet and Dormy [5], they established ill-posedness for the Cauchy problem of the linearized Prandtl equation around non-monotonic special solution which is independent of  $x$  and satisfies the heat equation. In [6] and [7], some nonlinear ill-posedness were established with this counterexample. Then it is natural to consider the problem that does this linear ill-posedness happen whenever the non-degenerate critical points appear. In this paper, we concern the linearized Prandtl equation around general stationary solutions with non-degenerate critical points depending on  $x$  which could be considered as the time-periodic solutions and show some ill-posedness.

*Keywords:* Prandtl Equations,

## 1 Introduction

The behavior of the solution to the vanishing viscosity limit of Navier-Stokes equation near a solid boundary is an outstanding open problem both in fluid mechanics and in mathematics. To describe this problem, let us consider the two-dimensional Navier-Stokes equation on a half-space:

$$\begin{aligned} \partial_t u^\nu + u^\nu \partial_x u^\nu + v^\nu \partial_y u^\nu + \partial_x p^\nu - \nu \Delta u^\nu &= 0, \\ \partial_t v^\nu + u^\nu \partial_x v^\nu + v^\nu \partial_y v^\nu + \partial_y p^\nu - \nu \Delta v^\nu &= 0, \\ \partial_x u^\nu + \partial_y v^\nu &= 0, \\ (u^\nu, v^\nu)|_{y=0} &= (0, 0), \end{aligned} \tag{1.1}$$

where  $(x, y) \in \mathbb{R} \times \mathbb{R}^+$ ,  $u^\nu$  is the tangential components of velocity to the boundary  $(x, 0)$ , and  $v^\nu$  is the normal components. A natural question is that does the solution

$(u^\nu, v^\nu)$  convergence to the solution of Euler equation:

$$\begin{aligned}
\partial_t u^E + u^E \partial_x u^E + v^E \partial_y u^E + \partial_x p^E &= 0, \\
\partial_t v^E + u^E \partial_x v^E + v^E \partial_y v^E + \partial_y p^E &= 0, \\
\partial_x u^E + \partial_y v^E &= 0, \\
v^\nu|_{y=0} &= 0,
\end{aligned} \tag{1.2}$$

when  $\nu \rightarrow 0$ . As there is the no-slip condition:  $u^\nu|_{y=0} = 0$  in Navier-Stokes equations, the transition from zero velocity at the boundary to the full magnitude at some distance from it take place in a very thin layer. Then the flow can be divided into two regions: the boundary layer where viscous friction plays an essential part, and the remaining region outside this layer where friction may be neglected (the outer Euler flow). It was seen from several exact solution of Navier-Stokes equations that the boundary-layer thickness is proportional to  $\sqrt{\nu}$ , therefore we may write  $(u^\nu, v^\nu)$  formally as:

$$\begin{aligned}
u^\nu(t, x, y) &= u^E(t, x, y) + u^B(t, x, \frac{y}{\sqrt{\nu}}), \\
v^\nu(t, x, y) &= v^E(t, x, y) + \sqrt{\nu} v^B(t, x, \frac{y}{\sqrt{\nu}}),
\end{aligned}$$

corresponding to the  $p^\nu(t, x, y)$ :

$$p^\nu(t, x, y) = p^E(t, x, y) + p^B(t, x, \frac{y}{\sqrt{\nu}}).$$

Denote  $Y = \frac{y}{\sqrt{\nu}}$ , let us define:

$$\begin{aligned}
u(t, x, Y) &:= u^E(t, x, 0) + u^B(t, x, Y), \\
v(t, x, Y) &:= Y \partial_y v^E(t, x, 0) + v^B(t, x, Y),
\end{aligned}$$

we formally obtain, from the Navier-Stokes equation by making  $\nu$  tend to zero, the following system:

$$\begin{aligned}
\partial_t u + u \partial_x u + v \partial_Y u - \partial_Y^2 u + \partial_x P &= 0, \\
\partial_x u + \partial_Y v &= 0, \\
(u, v)|_{Y=0} &= 0, \\
\lim_{Y \rightarrow \infty} u &= u^E(t, x, 0),
\end{aligned} \tag{1.3}$$

where the pressure  $p$  does not depend on  $Y$ , and satisfies the Bernoulli equation:

$$\partial_t u^E(t, x, 0) + u^E(t, x, 0) \partial_x u^E(t, x, 0) + \partial_x p = 0.$$

These are the Prandtl equations, proposed by Ludwig Prandtl [10] in 1904. Although the Prandtl equations have been simplified to a great extent, as compared with

the Navier-Stokes equations, they are still so difficult from the mathematical point of view that not very many general statements about them can be made.

For the steady version, the von Mises transformation reduces these equations to a degenerated parabolic equation under the condition  $u > 0$ . Let us consider the equation in the domain  $D = \{0 < x < X, 0 < Y < \infty\}$  with the condition

$$u(0, Y) = u_1(Y),$$

and introduce new independent variables by

$$\xi = x, \quad \psi = \psi(x, Y),$$

where

$$u = \partial_Y \psi, \quad v = \partial_x \psi, \quad \psi(x, 0) = 0,$$

and a new function  $\omega(\xi, \psi) = u^2(x, Y)$ . Then the domain  $D$  turns into  $\{0 < \xi < X, 0 < \psi < \infty\}$ , and the Prandtl system reduces to the equation:

$$\sqrt{\omega} \partial_\psi^2 \omega - \partial_\xi \omega - 2p_\xi = 0,$$

with the condition:

$$\omega(\xi, 0) = 0, \quad \omega(0, \psi) = \omega_1(\psi), \quad \lim_{\psi \rightarrow \infty} \omega(\xi, \psi) = U^2(x)$$

where

$$\omega_1(\int_0^Y u_1(\eta) d\eta) = u_1^2(Y).$$

Owing to this transformation, the maximum and comparison principles apply for the equation, and we get the existence and uniqueness of (1.3):

**Proposition 1.1.** ([9]) *Assume that  $u_1(y) > 0$  for  $y > 0$ ;  $u_1(0) = 0$ ,  $u_1'(0) > 0$ ,  $u_1(Y) \rightarrow U(0) \neq 0$  as  $Y \rightarrow \infty$ ;  $p_x$  is continuously differentiable on  $[0, X]$ ;  $u_1(Y)$ ,  $u_1'(Y)$ ,  $u_1''(Y)$  are Hölder continuous and bounded for  $0 \leq Y < \infty$ . Moreover, assume that for small  $y$  the compatibility condition is satisfied at the point  $(0, 0)$ :*

$$u_1''(Y) - p_x(0) = O(Y^2).$$

*Then, for some  $X > 0$  there exists a unique solution  $u(x, Y)$ ,  $v(x, Y)$  of the Prandtl equation (1.3) in  $D$ , which have the following properties:*

- (1)  $u(x, Y)$  is bounded and continuous in  $\overline{D}$ ,  $u > 0$  for  $Y > 0$ ;
- (2)  $\partial_Y u > m > 0$  for  $0 < Y \leq Y_0$ , where  $m$  and  $Y_0$  are constants;
- (3)  $\partial_Y u$  and  $\partial_Y^2 u$  are bounded and continuous in  $D$ ;
- (4)  $\partial_x u$ ,  $v$  and  $\partial_Y v$  are bounded and continuous in any finite portion of  $\overline{D}$ .

*Moreover, if  $|u_1'(Y)| \leq m_1 e^{-m_Y}$ , where  $m_1$  and  $m_2$  are positive constants, then  $\partial_x u$  and  $\partial_Y v$  are bounded in  $D$ . If  $p_x \leq 0$ , then such a solution exists in  $D$  for any  $X > 0$ .*

We refer to [9] for details. If  $p_x \leq 0$ , then  $X = \infty$ , that is the separation of boundary layer does not appear under this condition. When  $p_x > 0$ , the result is only valid local in  $x$ , and this leads eventually to boundary layer separation, see [1] [4].

The unsteady case is much complicated, and we have to imposed more conditions. Under the condition  $\partial_y u > 0$ , Crocco transformation, introduced by Crocco (1941), see [2], reduces the boundary layer system to a single quasilinear equation of the degenerate parabolic type and satisfies the maximum principle. In contrast to the von Mises transformation, it reduces the Prandtl system to an equation in a finite domain and the boundary condition become nonlinear. basing on this transformation, Oleinik and Samokhin [9] proved local in time well-posedness, and Xin and Zhang [13] proved the global in time well-posedness. Other positive mathematical results concern the case of analytic data. In [11] [12] Sammartino, M., and Caflisch, R.E. proved the short time existence and uniqueness when the data are analytic; this result was improved in [1] [8] where analyticity is required in the  $x$ -axis direction, while, using the regularizing effect of the viscosity.

In a recent result of Gérard-Varet and Dormy [5], they established ill-posedness for the Cauchy problem of the linearized Prandtl equation around non-monotonic special solution which is independent of  $x$  and satisfies the heat equation. In [6] and [7], some nonlinear ill-posedness were established with this counterexample. As the system is global well posed under the condition  $\partial_y u > 0$ . Then it is natural to consider the problem that does this linear ill-posedness happen whenever the non-degenerate critical points appear. As the well-posedness of stationary Prandtl equation has been established in [9] and the stationary solution could be considered as special time-periodic solution, and the time-periodic boundary layer system

$$\begin{aligned} \partial_t u + u \partial_x u + v \partial_y u + p_x &= \partial_y^2 u, \\ \partial_x u + \partial_y v &= 0 \end{aligned} \tag{1.4}$$

where  $(t, x, y) \in \mathbb{T} \times [0, X) \times \mathbb{R}^+$ , with the condition

$$u(t, x, 0) = v(t, x, 0) = 0, \quad u(t, 0, y) = u_1(t, y), \quad \lim_{y \rightarrow \infty} u(t, x, y) = U(t, x), \tag{1.5}$$

is well-posed under the condition  $\partial_y u > 0$ , see [9], then it is interesting to consider the linearized equation around stationary solution  $(u_0(x, y), v_0(x, y))$  where  $u_0$  has non-degenerate critical points, that is

$$\begin{aligned} \partial_t u + u_0 \partial_x u + v_0 \partial_y u + u \partial_x u_0 + v \partial_y u_0 - \partial_y^2 u &= 0, \\ \partial_x u + \partial_y v &= 0 \end{aligned} \tag{1.6}$$

where  $(t, x, y) \in \mathbb{T} \times [0, X) \times \mathbb{R}^+$ , with the condition

$$u(t, x, 0) = v(t, x, 0) = 0, \quad u(t, 0, y) = u_1(t, y), \quad \lim_{y \rightarrow \infty} u(t, x, y) = 0. \tag{1.7}$$

Let us introduce the following function spaces:

$$W_\alpha^{s, \infty} := \{f = f(y), \quad e^{\alpha y} f \in W^{s, \infty}(\mathbb{R}^+)\},$$

with the norm:  $\|f\|_{W_\alpha^{s,\infty}} = \|e^{\alpha y} f\|_{W^{s,\infty}},$

$$H_\beta^m := H^m(\mathbb{T}_t, W_\beta^{0,\infty}(\mathbb{R}_y^+)),$$

and

$$\overline{H}_\beta^m := H_\beta^m \cap C^1(\mathbb{T}_t, W_\beta^{2,\infty}(\mathbb{R}_y^+)).$$

Our main result then reads:

**Theorem 1.1.** *Let  $u_0 - U \in C^0([0, X_0]; W_\alpha^{4,\infty}(\mathbb{R}^+)) \cap C^1([0, X_0]; W_\alpha^{2,\infty}(\mathbb{R}^+))$ ,  $u_0|_{x=0}$  has a non-degenerate critical point. If there exists  $X > 0$ , such that for every  $u_1(t, y) \in \overline{H}_\beta^m$ , with  $\beta < \alpha$ , equations (1.6) (1.7) have a unique solution, let us denote  $u(x, \cdot) := \mathfrak{X}(x, \xi)u_1$ , where  $u(x, \cdot)$  is the solution of (1.6) and (1.7) with  $u|_{x=\xi} = u_1$ , then there exist a  $\delta > 0$ , such that for every  $\epsilon > 0$ ,*

$$\sup_{0 \leq \xi \leq x \leq \epsilon} \|e^{-\delta(x-\xi)} \sqrt{|\partial_t|} \mathfrak{X}(x, \xi) \|_{\mathcal{L}(H_\beta^m, H_\beta^{m-\sigma})} = \infty, \quad \forall m \geq 0, \sigma \in [0, \frac{1}{2}).$$

If  $\lim_{y \rightarrow \infty} u_0(x, y) = C$ , where  $C \geq 0$  is a constant, the result is valid for  $\alpha = \beta$ .

By modifying the construction of approximate solution performed in [5], we construct an unstable quasimode which is based on an asymptotic analysis of (1.6) in the high time frequency limit and get the ill-posedness. We also need the following lemma:

**Lemma 1.2.**  *$C > 0$  is a real constant, there exists  $\tau \in \mathbb{C}$ , with  $\text{im}\tau < 0$ , and a solution  $W = W(z)$  of*

$$(\tau - z^2)^2 \frac{d}{dz} W + iC \frac{d^3}{dz^3} ((\tau - z^2)W) = 0,$$

such that

$$\lim_{z \rightarrow -\infty} W = 0, \quad \lim_{z \rightarrow +\infty} W = 1.$$

The only difference between this lemma and the the "spectral condition" in [5] is that there is a constance  $C$  in this lemma. And this lemma could be proved by considering the eigenvalue problem:

$$\frac{1}{z^2 + 1} u'' + \frac{6z}{(z^2 + 1)^2} u' + \frac{6}{(z+1)^2} u = \frac{\alpha}{C} u,$$

for  $z \in R$ , then extend  $z$  from  $R$  to  $\mathbb{C}$ , by the theory of ordinary differential equation and the complex change of variable, we get the existence of  $W$ . We refer to [5] for details of the proof and list some property of  $W$ :

$$|W(z) - 1| \leq C' e^{-c|z|^2},$$

when  $z > 0$ ,

$$|W(z)| \leq C' e^{-c|z|^2},$$

when  $z < 0$ , and

$$W^{(k)}(z) = O(e^{-c|z|^2}), \quad z \rightarrow \infty,$$

where the constants  $C'$ ,  $c > 0$ .

**Remark 1.1.** *Actually, the time-periodic ill-posedness of the equation correspondent to the large-time instability of the equation. As in Z.Xin and L.Zhang [13], it is well posed in global time with the condition  $\partial_y u > 0$ . Therefore there is some essential difference when the points satisfying  $\partial_y u = 0$ ,  $\partial_y u \neq 0$  and depending on  $x$  appear.*

## 2 The Proof of The Result

Firstly, let us construct the approximate solution by modifying the construction of D.Gérard-Varet and E.Dormy. As equation (1.6) has constant coefficients in  $t$ , a Fourier analysis can be performed, then we could look for solution in form

$$u(t, x, y) = e^{-ikt} \hat{u}^k, \quad v(t, x, y) = e^{-ikt} \hat{v}^k.$$

We can also separate corresponding frequency and amplitude in  $x$  from  $\hat{u}^k$ , which lead the ill-posedness. That is we can look for the approximate solution in the form:

$$u_k(t, x, y) = e^{-it - i\omega(k)x} u^k(x, y).$$

We denote  $\varepsilon := \frac{1}{k}$  and concern the case  $\varepsilon \ll 1$ .

Denote  $a$  the non-degenerate critical point of  $u_0|_{x=0}$ . Assume that  $\partial_y^2 u_0(0, a) < 0$ , then there exists  $0 < X_1 < X_0$ ,  $a(x) \in C^1([0, X_1])$ , such that

$$\partial_y u_0(x, a(x)) = 0, \quad \partial_y^2(x, a(x)) < 0.$$

In fact,  $a(x)$  satisfies the differential equation:

$$\partial_{xy} u_0(x, a(x)) + \partial_y^2 u_0(x, a(x)) a'(x) = 0,$$

with the condition  $a(0) = a$ .

Let

$$u_\varepsilon = -ie^{-i\frac{t}{\varepsilon} - i\frac{1}{\varepsilon} \int_0^x \omega(\varepsilon, \xi) d\xi} \frac{\partial_y(v_\varepsilon^{reg} + v_\varepsilon^{sl})}{\omega(\varepsilon, x)}, \quad (2.1)$$

$$v_\varepsilon = \frac{1}{\varepsilon} e^{-i\frac{t}{\varepsilon} - i\frac{1}{\varepsilon} \int_0^x \omega(\varepsilon, \xi) d\xi} (v_\varepsilon^{reg} + v_\varepsilon^{sl}) + ie^{-i\frac{t}{\varepsilon} - i\frac{1}{\varepsilon} \int_0^x \omega(\varepsilon, \xi) d\xi} \left( \frac{v_\varepsilon^{reg} + v_\varepsilon^{sl}}{\omega(\varepsilon, x)} \right)_x, \quad (2.2)$$

where

$$\omega(\varepsilon, x) = \frac{1}{-u_0(x, a(x)) + \sqrt{\frac{\varepsilon}{2}} |\partial_y^2 u_0(x, a(x))|^{\frac{1}{2}} \tau},$$

the "regular" velocity:

$$v_\varepsilon^{reg}(x, y) := H(y - a(x)) [u_0(x, y) - u_0(x, a(x)) + \sqrt{\frac{\varepsilon}{2}} |\partial_y^2 u_0(x, a(x))|^{\frac{1}{2}} \tau], \quad (2.3)$$

the shear layer velocity:

$$v_\varepsilon^{sl}(x, y) := \varphi(y - a(x)) \sqrt{\frac{\varepsilon}{2}} |\partial_y^2 u_0(x, a(x))|^{\frac{1}{2}} V\left[\left(\frac{|\partial_y^2 u_0(x, a(x))|}{2\varepsilon}\right)^{\frac{1}{4}} (y - a(x))\right], \quad (2.4)$$

$\varphi$  is a smooth truncation function near 0,

$$V(z) = (\tau - z^2)(W(z) - H(z)),$$

$W$  satisfies the equation:

$$(\tau - z^2)^2 \frac{d}{dz} W + i u_0(0, a) \frac{d^3}{dz^3} ((\tau - z^2)W) = 0, \quad (2.5)$$

with the condition

$$\lim_{z \rightarrow -\infty} W = 0, \quad \lim_{z \rightarrow +\infty} W = 1,$$

and  $H$  is the Heaviside function. In the expression of  $(u_\varepsilon, v_\varepsilon)$ , the "regular" velocity  $v_\varepsilon^{reg}$  is the main part of the approximate solution which satisfies the condition

$$u_\varepsilon|_{y=0} = 0,$$

and the shear layer velocity  $v_\varepsilon^{sl}$  is the modifying part as both  $v_\varepsilon^{reg}$  and its second derivative have jumps at  $y = a(x)$ .

It is easy to check that  $u_\varepsilon$  is analytic in  $t$ , and  $W_\beta^{2,\infty}$  in  $x, y$ , and

$$C_1 e^{\frac{\delta_0 t}{\sqrt{\varepsilon}}} \leq \left\| \frac{u_\varepsilon(t, x, y)}{e^{-i\frac{t}{\varepsilon}}} \right\|_{W_\beta^{2,\infty}} \leq C_2 \frac{1}{\varepsilon^{\frac{1}{4}}} e^{\frac{\delta_0 t}{\sqrt{\varepsilon}}}, \quad (2.6)$$

where  $C_1$ ,  $C_2$  and  $\delta_0$  are independent of  $\varepsilon$ .

Inserting  $u_\varepsilon, v_\varepsilon$  into (1.6), we have:

$$\begin{aligned} \partial_t u_\varepsilon + u_0 \partial_x u_\varepsilon + v_0 \partial_y u_\varepsilon + u_\varepsilon \partial_x u_0 + v_\varepsilon \partial_y u_0 - \partial_y^2 u_\varepsilon &= I_\varepsilon, \\ \partial_x u_\varepsilon + \partial_y v_\varepsilon &= 0, \end{aligned}$$

moreover,  $u_\varepsilon, v_\varepsilon$  satisfy the condition:

$$(u_\varepsilon, v_\varepsilon)|_{y=0} = (0, 0), \quad \lim_{y \rightarrow \infty} u_\varepsilon = 0,$$

where

$$\begin{aligned} I_\varepsilon = & e^{-i\frac{t}{\varepsilon} - i\frac{1}{\varepsilon} \int_0^x \omega(\varepsilon, \xi) d\xi} \left[ -\frac{1}{\varepsilon} \frac{\partial_y (v_\varepsilon^{reg} + v_\varepsilon^{sl})}{\omega(\varepsilon, x)} - \frac{u_0}{\varepsilon} \partial_y (v_\varepsilon^{reg} + v_\varepsilon^{sl}) - i u_0 \partial_x \left( \frac{\partial_y (v_\varepsilon^{reg} + v_\varepsilon^{sl})}{\omega(\varepsilon, x)} \right) \right. \\ & - i v_0 \frac{\partial_y^2 (v_\varepsilon^{reg} + v_\varepsilon^{sl})}{\omega(\varepsilon, x)} - i \partial_x u_0 \frac{\partial_y (v_\varepsilon^{reg} + v_\varepsilon^{sl})}{\omega(\varepsilon, x)} + \frac{1}{\varepsilon} \partial_y u_0 (v_\varepsilon^{reg} + v_\varepsilon^{sl}) \\ & \left. + i \partial_y u_0 \partial_x \left( \frac{v_\varepsilon^{reg} + v_\varepsilon^{sl}}{\omega(\varepsilon, x)} \right) + i \frac{\partial_y^3 (v_\varepsilon^{reg} + v_\varepsilon^{sl})}{\omega(\varepsilon, x)} \right] \end{aligned}$$

For  $y \neq a(x)$ ,  $I_\varepsilon$  could be write as

$$I_\varepsilon = e^{-i\frac{t}{\varepsilon} - i\frac{1}{\varepsilon} \int_0^x \omega(\varepsilon, \xi) d\xi} (I_1 + I_2 + I_3),$$

where

$$\begin{aligned}
I_1 &= -iu_0\partial_x\left(\frac{\partial_y(v_\varepsilon^{reg} + v_\varepsilon^{sl})}{\omega(\varepsilon, x)}\right) - iv_0\frac{\partial_y^2(v_\varepsilon^{reg} + v_\varepsilon^{sl})}{\omega(\varepsilon, x)} - i\partial_x u_0\frac{\partial_y(v_\varepsilon^{reg} + v_\varepsilon^{sl})}{\omega(\varepsilon, x)} \\
&\quad + i\partial_y u_0\partial_x\left(\frac{v_\varepsilon^{reg} + v_\varepsilon^{sl}}{\omega(\varepsilon, x)}\right), \\
I_2 &= -\frac{1}{\varepsilon}\left(\frac{1}{\omega(\varepsilon, x)} + u_0(x, y)\right)\partial_y v_\varepsilon^{reg} + \frac{1}{\varepsilon}\partial_y u_0 v_\varepsilon^{reg} + i\frac{\partial_y^3 v_\varepsilon^{reg}}{\omega(\varepsilon, x)} \\
&= i\frac{\partial_y^3 v_\varepsilon^{reg}}{\omega(\varepsilon, x)},
\end{aligned}$$

and

$$I_3 = -\frac{1}{\varepsilon}\left(\frac{1}{\omega(\varepsilon, x)} + u_0(x, y)\right)\partial_y v_\varepsilon^{sl} + \frac{1}{\varepsilon}\partial_y u_0 v_\varepsilon^{sl} + i\frac{\partial_y^3 v_\varepsilon^{sl}}{\omega(\varepsilon, x)}.$$

As  $u_0$  satisfies the equation

$$u_0\partial_x u_0 - \int_0^y \partial_x u_0 dy' \partial_y u_0 + p_x = \partial_y^2 u_0,$$

by solving the ordinary equation with respect to  $y$ ,  $v_0(\cdot, y)$  could be write as:

$$v_0(\cdot, y) = -u_0 \int_0^y (\partial_0^2 u_0 - p_x) dy',$$

and

$$\partial_x u_0(\cdot, y) = \partial_y u_0 \int_0^y (\partial_y^2 u_0 - p_x) dy' + u_0(\partial_y^2 u_0 - p_x),$$

then it is easy to check that

$$\partial_x u_0(\cdot, y) = O(1), \quad v_0(\cdot, y) = O(y), \quad \text{as } y \rightarrow \infty,$$

if  $p_x \equiv 0$  ( $U(x) \equiv C$ ), we have

$$\partial_x u_0(\cdot, y) = O(e^{-\alpha y}), \quad v_0(\cdot, y) = O(1), \quad \text{as } y \rightarrow \infty.$$

Then by the property of  $(u_0, v_0)$ , we have

$$\|I_1(x, \cdot)\|_{W_\beta^{0, \infty}} \leq C_3,$$

where  $C_3$  is a constant independent of  $\varepsilon$ .



As

$$\begin{aligned}
I_3 &= -\frac{1}{\varepsilon} \left( \frac{1}{\omega(\varepsilon, x)} + u_0(x, y) \right) \partial_y v_\varepsilon^{sl} + \frac{1}{\varepsilon} \partial_y u_0 v_\varepsilon^{sl} + i \frac{\partial_y^3 v_\varepsilon^{sl}}{\omega(\varepsilon, x)} \\
&= -\frac{1}{\varepsilon} [u_0(x, y) - u_0(x, a(x)) - \partial_y u_0(x, a(x))(y - a(x)) \\
&\quad - \frac{1}{2} \partial_y^2 u_0(x, a(x))(y - a(x))^2] \partial_y v_\varepsilon^{sl} \\
&\quad + \frac{1}{\varepsilon} [\partial_y u_0(x, y) - \partial_y u_0(x, a(x)) - \partial_y^2 u_0(x, a(x))(y - a(x))] v_\varepsilon^{sl} \\
&\quad + \left\{ -\frac{1}{\varepsilon} \left[ \frac{1}{2} \partial_y^2 u_0(x, a(x))(y - a(x))^2 + \sqrt{\frac{\varepsilon}{2}} |\partial_y^2 u_0(x, a(x))|^{\frac{1}{2}} \tau \right] \partial_y v_\varepsilon^{sl} \right. \\
&\quad \left. + \frac{1}{\varepsilon} \partial_y^2 u_0(x, a(x))(y - a(x)) v_\varepsilon^{sl} - i u_0(0, a) \partial_y^3 v_\varepsilon^{sl} \right\} \\
&\quad + i [u_0(0, a) - u_0(x, a(x)) + \sqrt{\frac{\varepsilon}{2}} |\partial_y^2 u_0(x, a(x))|] \partial_y^3 v_\varepsilon^{sl} \\
&:= I_{31} + I_{32} + I_{33} + I_{34}.
\end{aligned}$$

Set  $z = (\frac{|\partial_y^2 u_0(x, a(x))|}{2\varepsilon})^{\frac{1}{4}}(y - a(x))$ , it is easy to check that:

$$\|I_{31}(x, \cdot)\|_{W_\beta^{0,\infty}} + \|I_{32}(x, \cdot)\|_{W_\beta^{0,\infty}} \leq C_4,$$

where  $C_4$  is a constant independent of  $\varepsilon$ .

$$\begin{aligned}
I_{33} &= \frac{|\partial_y^2 u_0(x, a(x))|^{\frac{1}{2}}}{(2\varepsilon)^{\frac{1}{2}}} (z^2 - \tau) \partial_y v_\varepsilon^{sl} - 2 \left( \frac{|\partial_y^2 u_0(x, a(x))|}{2\varepsilon} \right)^{\frac{3}{4}} z v_\varepsilon^{sl} \\
&= \left( \frac{|\partial_y^2 u_0(x, a(x))|}{2\varepsilon} \right)^{\frac{3}{4}} \sqrt{\frac{\varepsilon}{2}} |\partial_y^2 u_0(x, a(x))|^{\frac{1}{2}} \varphi(y - a(x)) ((z^2 - \tau)V' - 2zV \\
&\quad - i u_0(0, a) \partial_y^3 V''') + \varphi'(y - a(x)) (c_1 \varepsilon^{k_1} V + c_2 \varepsilon^{k_2} V'') + \varphi''(y - a(x)) c_3 \varepsilon^{k_3} V' \\
&\quad + \varphi'''(y - a(x)) c_4 \varepsilon^{k_4} V,
\end{aligned}$$

inserting the expression of  $V$ , we have:

$$(z^2 - \tau)V' - 2zV - i u_0(0, a) V''' = 0,$$

when  $z \neq 0$ , and choosing  $\varphi$ , such that the derivative of  $\varphi(y - a(x))$  is 0 near  $y = a(x)$ , as  $V$  and its derivative decreases exponentially, then we have

$$\|I_{33}(x, \cdot)\|_{W_\beta^{0,\infty}} \leq C_5,$$

where  $C_4$  is a constant independent of  $\varepsilon$ .

By the regularity of  $u_0(x, y)$ , we have

$$\|I_{34}(x, \cdot)\|_{W_\beta^{0,\infty}} \leq C_6,$$

when  $0 < x < \varepsilon^{\frac{1}{4}}$ , where  $C_6$  is a constant independent of  $\varepsilon$ . Denote  $J_\varepsilon(x, y) = e^{i\frac{t}{\varepsilon}} I_\varepsilon(t, x, y)$ , Then we have

$$\|J_\varepsilon(x, \cdot)\|_{W_\beta^{0,\infty}} \leq C_7 e^{\frac{\delta_0 x}{\sqrt{\varepsilon}}}, \quad (2.7)$$

when  $0 < x < \varepsilon^{\frac{1}{4}}$ , where  $C_7$  is a constant independent of  $\varepsilon$ .

Let us assume that (1.6) has a unique solution and  $\forall \delta > 0, \exists \epsilon_0 > 0, m \geq 0$ , and  $0 \leq \sigma < \frac{1}{2}$ ,

$$\sup_{0 \leq \xi \leq x \leq \epsilon_0} \|e^{-\delta(x-\xi)} \sqrt{|\partial_t|} \mathfrak{X}(x, \xi) \|_{\mathcal{L}(H^m, H^{m-\sigma})} \leq C_8, \quad (2.8)$$

Let  $\mathfrak{X}_\varepsilon(x, \xi)$  be the restriction of  $\mathfrak{X}(x, \xi)$  to the tangential Fourier mode  $\frac{1}{\varepsilon}$ , we have

$$\|\mathfrak{X}_\varepsilon(x, \xi)\|_{\mathcal{L}(W_\beta^{0,\infty})} \leq C_8 \varepsilon^{-\sigma} e^{\frac{\delta(x-\xi)}{\sqrt{\varepsilon}}}. \quad (2.9)$$

Let us write the first equation of (1.6) as

$$\begin{aligned} & \partial_x u - \frac{\partial_x u_0}{u_0} \int_0^y \partial_x u dy' + \frac{\partial_t u}{u_0} + \frac{\partial_x u_0}{u_0} u + \frac{v_0}{u_0} \partial_y u - \partial_y^2 u \\ := & \partial_x u - \frac{\partial_x u_0}{u_0} \int_0^y \partial_x u dy' + Lu, \end{aligned}$$

and denote  $L_\varepsilon$  the restriction of  $L$  to the tangential Fourier mode  $\frac{1}{\varepsilon}$ . Let  $U(x, y)$  be the solution of

$$\partial_x U - \frac{\partial_x u_0}{u_0} \int_0^y \partial_x U dy' + L_\varepsilon U = 0,$$

with the condition  $U(x, y) = e^{i\frac{t}{\varepsilon}} u_\varepsilon(t, x, y)$  when  $x = 0$ . Then

$$\|U(x, y)\|_{W_\beta^{0,\infty}} \leq C_9 \varepsilon^{-\sigma} e^{\frac{\delta x}{\sqrt{\varepsilon}}}. \quad (2.10)$$

Then  $\tilde{U} = U - e^{i\frac{t}{\varepsilon}} u_\varepsilon$  satisfies:

$$\partial_x \tilde{U} - \frac{\partial_y u_0}{u_0} \int_0^y \partial_x \tilde{U} dy' + L_\varepsilon \tilde{U} = J_\varepsilon, \quad (2.11)$$

Assuming that the unique solution of (2.11) has the form  $\int_0^x \mathfrak{X}_\varepsilon(x, \xi) Q_\varepsilon(\xi, \cdot) d\xi$ , then we have

$$Q_\varepsilon(x, y) - \frac{\partial_y u_0}{u_0} \int_0^y Q_\varepsilon(x, y') dy' = J_\varepsilon(x, y),$$

by solving the differential equation we obtain

$$Q_\varepsilon(x, y) = J_\varepsilon(x, y) + \partial_y u_0 \int_0^y J_\varepsilon(x, y') u_0(x, y') dy'.$$

It is easy to check that  $\|Q_\varepsilon(x, \cdot)\|_{W_\beta^{0,\infty}} \leq C_{10}$ , where  $C_{10}$  is a constant independent of  $\varepsilon$ . Then  $\tilde{U} = \int_0^x \mathfrak{X}_\varepsilon(x, \xi) Q_\varepsilon(x, \xi) d\xi$ , we have

$$\begin{aligned} \|\tilde{U}(x, \cdot)\|_{W_\beta^{0,\infty}} &= \sup_{y>0} |e^{\beta y} \int_0^x \mathfrak{X}_\varepsilon(x, \xi) Q_\varepsilon(\xi, y) d\xi| \\ &\leq \int_0^x \sup_{y>0} |e^{\beta y} \mathfrak{X}_\varepsilon(x, \xi) Q_\varepsilon(\xi, y)| d\xi \\ &= \int_0^x \|\mathfrak{X}_\varepsilon(x, \xi) Q_\varepsilon(\xi, y)\|_{W_\beta^{0,\infty}} d\xi \\ &\leq \int_0^x C_8 \varepsilon^{-\sigma} e^{\frac{\delta(x-\xi)}{\sqrt{\varepsilon}}} \leq C_{11} \varepsilon^{\frac{1}{2}-\sigma} e^{\frac{\delta_0 x}{\sqrt{\varepsilon}}}, \end{aligned}$$

therefore

$$\|U(x, \cdot)\|_{W_\beta^{0,\infty}} \geq \|e^{i\frac{x}{\varepsilon}} u_\varepsilon(x, \cdot)\|_{W_\beta^{0,\infty}} - \|\tilde{U}(x, \cdot)\|_{W_\beta^{0,\infty}} \geq C_{12} e^{\frac{\delta_0 x}{\sqrt{\varepsilon}}}, \quad (2.12)$$

when  $\delta < \delta_0$ . Then (2.12) contradicts (2.10), as soon as  $\frac{\sigma}{\delta_0 - \delta} |\ln \varepsilon| \sqrt{\varepsilon} \ll x < \varepsilon^{\frac{1}{4}}$ .

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